CH 7 – BREAK-EVEN POINT, LINEAR FUNCTIONS

☐ INTRODUCTION – BUSINESS TERMS

Certainly the ultimate goal of a business is not merely to "break even"; however, the break-even point is one of the most important concepts in business. It tells the business owner the point (in production, time, or investments) where losses have probably ended and profits will begin to appear (or, unfortunately, the other way around). Every business which requires funding (either from investments, like selling stock, or the borrowing of money) requires a written business plan stating the projected break-even time.

A few business terms, with simplified definitions, will help us understand how algebra can represent the real world. We'll use the term ***Revenue*** to represent all the money that a company takes in through sales and services. The term ***Cost*** represents the money spent by the company to produce those sales and services. And we'll define ***Profit*** to be the <u>difference</u> between revenue and cost. We're ready to write a formula now:

Profit = Revenue – Cost

$$\boxed{P = R - C}$$

☐ *REVIEW OF INEQUALITIES*

The fact that 3 is **less than** 5 is written

$$3 < 5$$

If we want to say that x is **less than or equal to** 5, we write

$$x \leq 5$$

Similarly, we might write $10 > 8$ to say that 10 is **greater than** 8, and we write $n \geq 15$ to say that n is **greater than or equal to** 15. Notice that $7 \geq 7$ is a true statement because 7 <u>is</u> greater than or equal to 7. (Namely, it's equal to 7.)

Homework

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1. a. Solve the profit formula $P = R - C$ for R.

   b. Solve the profit formula $P = R - C$ for C.

2. True/False:

   a. $7 < 9$　b. $-2 < 7$　c. $-3 < -7$　d. $-9 < -1$

   e. $8 \geq 8$　f. $0 > 9$　g. $0 > -9$　h. $-2 \leq -2$

   i. $-2 < -2$　j. $0 < 0$　k. $0 \geq 0$　l. $-12 > -5$

   m. $\pi > \sqrt{2}$　n. $2\pi \geq \pi$　o. $\sqrt{2} < \sqrt{3}$　p. $\sqrt{10} < \pi$

3. Consider the inequality $n \geq 7$. Which of the following values of n would make the statement true?

   a. 15　b. 7.001　c. 6.999　d. 7　e. 2　f. -3　g. -29　h. $\sqrt{40}$

4. Which of the following values of x will satisfy the inequality $x < -3$?

a. -100 b. -3.001 c. -3 d. -2.999 e. 0 f. 3.14 g. 140

❑ *DEFINITION OF BREAK-EVEN*

There are two ways to define the ***break-even point***. One is to say that break-even occurs when revenues match cost; that is, when $R = C$. On the other hand, if the revenues and cost are the same, then there's zero profit. (See that?) So break-even can also be defined as the point at which the profit is 0.

Either equation can be used to find the ***break-even point***:

$$R = C$$

$$P = 0$$

To prove that the profit must be zero when revenue = cost, we can use a little algebra. Assume that

$R = C$ (one criterion for break-even)

$\Rightarrow R - C = 0$ (subtract C from each side of the equation)

$\Rightarrow P = 0$ (substitute, since $P = R - C$)

❑ USING A TABLE

Assuming w represents the number of widgets a company manufactures and sells, let's assume the formula for revenue to be:

$$R = 4w$$

And a cost formula to go with it:

$$C = 2w + 8$$

The Revenue formula might come from the fact that we sell our widgets for $4 apiece.

The Cost formula may be the result of the fact that each widget costs $2 to manufacture, together with a fixed cost (salaries, rent, utilities, etc.) of $8.

Let's create a table with revenue, cost, and profit for various quantities of widgets sold, according to the given formulas:

w	$R = 4w$	$C = 2w + 8$	$P = R - C$
Widgets	Revenue	Cost	Profit
0	$0	$8	–$8
1	$4	$10	–$6
2	$8	$12	–$4
3	$12	$14	–$2
4	**$16**	**$16**	**$0**
5	$20	$18	$2
6	$24	$20	$4

Break-Even Point →

Now for the important observations:

- ✓ As the number of widgets increases, so do the revenue and the cost. Also, the profit increases; that is, the numbers –8, –6, –4, –2, 0, 2, and 4 are getting larger.
- ✓ If we produce and sell anywhere from 0 to 3 widgets, we incur a loss (a negative profit). That is, when $w \le 3$, $P < 0$.

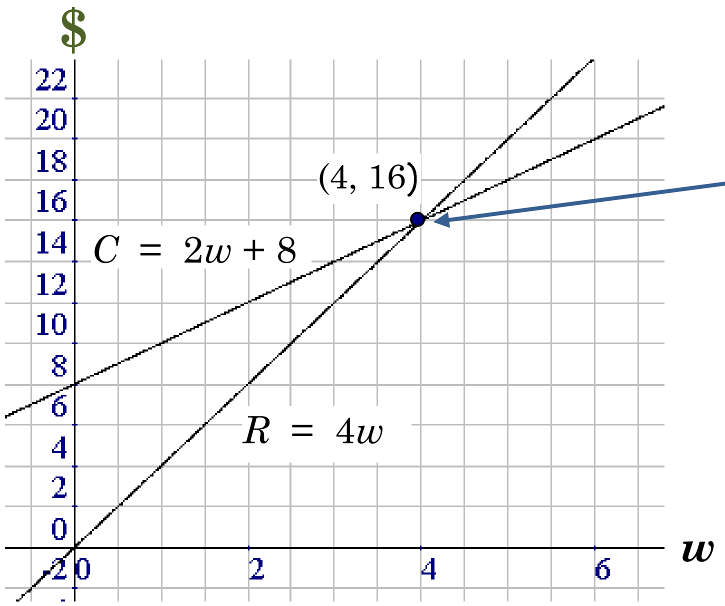
- ✓ At a production level of 4 widgets we break even. This is the point at which the revenue ($16) equals the cost ($16). But also note that this is where the profit is $0. Either way we look at it, according to the two definitions, <u>$w = 4$ is the break-even point</u>. When $w = 4$, $P = 0$.

- ✓ Any number of widgets beyond 4 (that is, 5 or more) produces a positive profit (we're making money!). Thus, if $w \geq 5$, then $P > 0$.

- ✓ In summary, a **loss** occurs when $w \leq 3$, the **break-even point** occurs when $w = 4$, and a **profit** results when $w \geq 5$.

❑ USING A GRAPH

We're going to make a picture of this situation of revenue, cost, profit, and the break-even point. Our grid will contain the graphs of both the revenue and cost formulas from the previous example. Note that the horizontal axis will certainly be w, the number of widgets, but the vertical axis will be the generic category money, since both revenue and cost are in units of money.



The ***break-even point*** occurs when $w = 4$, where both the revenue and the cost are $16.

Fewer than 4 widgets results in a loss — more than 4 produces a profit.

☐ *USING AN EQUATION*

The table and the graph were quite useful in determining and understanding the relationship among revenue, cost, profit, and break-even. But these take time to construct and they may give us only approximations. Let's use algebra to solve the same problem for the third time.

<u>EXAMPLE 1:</u> **The revenue formula is $R = 4w$ and the cost formula is $C = 2w + 8$. Find the break-even point.**

<u>Solution:</u> We recall that the break-even point occurs when the revenue equals the cost. Thus,

$R = C$ (to calculate break-even)

$4w = 2w + 8$ (substituting the given formulas)

$4w - \mathbf{2w} = 2w - \mathbf{2w} + 8$ (subtract $2w$ from each side)

$2w = 8$ (simplify each side)

$\frac{2w}{\mathbf{2}} = \frac{8}{\mathbf{2}}$ (divide each side by 2)

$w = 4$ (simplify)

Thus, the break-even point is $\boxed{\text{4 widgets}}$

just as we saw with the table and the graph.

EXAMPLE 2: **The revenue formula is $R = 12w - 20$ and the cost formula is $C = 2w + 30$.**

a) Calculate the profit formula.

b) Find the break-even point using the profit formula.

Solution: a) The profit formula is $P = R - C$, so we can calculate the profit like this:

$P = R - C$ (the profit formula)

$\Rightarrow$ $P = (12w - 20) - (2w + 30)$ (notice the parentheses!)

$\Rightarrow$ $P = 12w - 20 - 2w - 30$ (distribute)

$\Rightarrow$ $\boxed{P = 10w - 50}$ (combine like terms)

For part b) we find the break-even point by setting the profit formula to 0:

$P = 0$ (one criterion for break-even)

$\Rightarrow$ $10w - 50 = 0$ (substitute the given profit formula)

$\Rightarrow$ $10w = 50$ (add 50 to each side of the equation)

$\Rightarrow$ $\boxed{w = 5}$ (divide each side of the equation by 10)

Homework

5. Regarding Example 2, part b), use the profit formula to show that we incur a loss if $w < 5$ and we enjoy a profit if $w > 5$. (A couple of examples will suffice.)

6. Suppose revenue and cost formulas are given by

$$R = 3w + 1 \qquad C = w + 5$$

a. Construct a table with columns for Widgets, Revenue, Cost, and Profit. Let w take the values from 0 to 4.

b. Use the table to determine the break-even point. Explain in <u>two</u> different ways how you arrived at your conclusion.

c. Graph both formulas on the same grid. Use the graphs to determine the break-even point.

d. Now find the break-even point by solving the formula equating Revenue and Cost.

7. Find the **break-even point** for the given revenue and cost formulas by solving an equation:

a. $R = 10w$ $C = 7w + 18$

b. $R = 5w + 1$ $C = 4w + 50$

c. $R = 8w - 9$ $C = 3w + 6$

d. $R = 72w + 12$ $C = 50w + 100$

8. Find the **profit** formula for the given revenue and cost formulas:

a. $R = 30w + 90$ $C = 22w - 13$

b. $R = 22w - 5$ $C = 10w + 17$

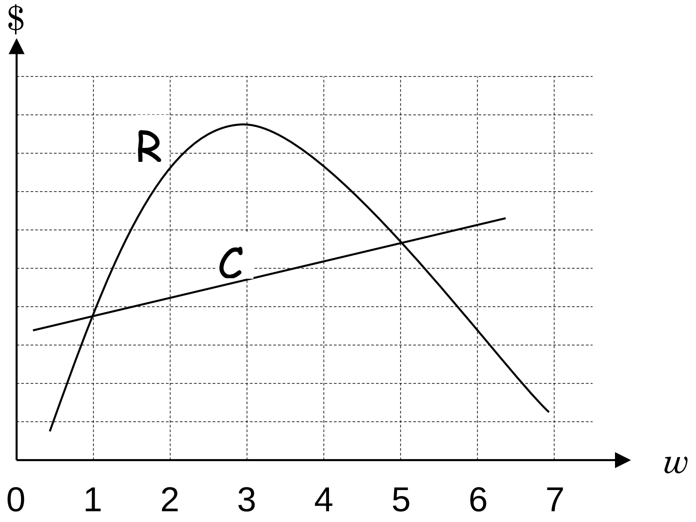
c. $R = w + 10$ $C = 8w - 14$

d. $R = 13w$ $C = 2w - 5$

e. $R = 99w + 17$ $C = 9w$

f. $R = 8w - 1$ $C = 7w - 10$

9. Consider the graphs of the revenue and cost formulas:

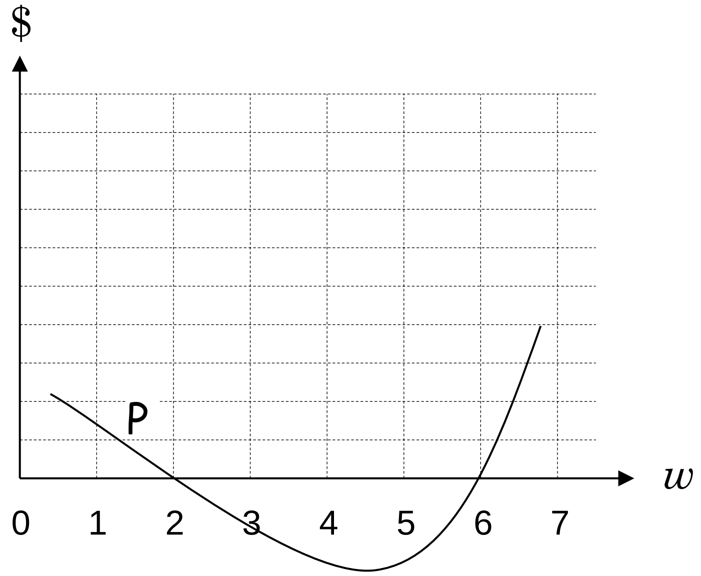


a. Find the two break-even points.

b. Is there a profit or loss when $w = 4$?

c. Is there a profit or loss when $w = 6$?

10. Consider the graph of the **profit** formula:



Find the two break-even points.

11. Find the **break-even point** given each profit formula:

    a. $P = 18w - 810$ b. $P = 2.5w - 300$

    c. $P = 5.23w - 287.65$ d. $P = -7.6w + 410.4$

Practice Problems

12. Let the revenue and cost formulas be given by $R = 7w + 1$ and $C = 5w + 11$. Construct a table with columns for widgets, revenue, cost, and profit, and let w take on the values from 3 to 6. Use the table to find the break-even point. Explain in two different ways how you arrived at your conclusion.

13. Use algebra (that is, solve an equation) to find the break-even point if the revenue and cost formulas are given by $R = 15w - 13$ and $C = 7w + 43$.

14. Find the profit formula if the revenue and cost formulas are given by $R = 10w + 13$ and $C = 6w - 5$.

15. Find the break-even point for the profit formula $P = 2.5w - 247.5$.

16. Graph $R = 3w - 4$ and $C = w + 1$ on the same grid, and then estimate the break-even point. How did you find that break-even point?

Solutions

1. a. $R = P + C$ b. $C = R - P$

2. a. T b. T c. F d. T e. T f. F
g. T h. T i. F j. F k. T l. F
m. T n. T o. T p. F

3. a, b, d **4.** a, b

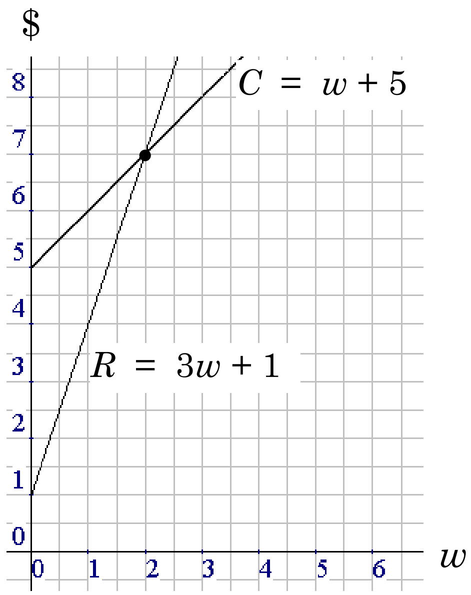
5. Suppose w is 3; then $P = 10(3) - 50 = 30 - 50 = -20$, a loss.
If w is 8, then $P = 10(8) - 50 = 80 - 50 = 30$, a profit.

6. a.

Widgets	Revenue	Cost	Profit
0	\$1	\$5	–\$4
1	\$4	\$6	–\$2
2	\$7	\$7	\$0
3	\$10	\$8	\$2
4	\$13	\$9	\$4

b. The break-even point is $w = 2$. It's the point where Revenue = Cost, and it's also the point where Profit = 0.

c.



d. The break-even point can be found by setting Revenue to Cost:

$R = C$ (to calculate break-even)
$3w + 1 = w + 5$ (substituting the given formulas)
$3w - \boldsymbol{w} + 1 = w - \boldsymbol{w} + 5$ (subtract w from each side)
$2w + 1 = 5$ (simplify)
$2w + 1 - \mathbf{1} = 5 - \mathbf{1}$ (subtract 1 from each side)
$2w = 4$ (simplify)
$\boldsymbol{w = 2}$ (divide each side by 2)

7. a. $w = 6$ b. $w = 49$ c. $w = 3$ d. $w = 4$

8. a. $P = R - C = (30w + 90) - (22w - 13) = 30w + 90 - 22w + 13$
$= 8w + 103$
b. $P = (22w - 5) - (10w + 17) = 22w - 5 -10w - 17 = 12w - 22$
c. $P = -7w + 24$
d. $P = 11w + 5$
e. $P = 90w + 17$
f. $P = w + 9$

9. a. $w = 1$ and $w = 5$ (found by looking at the w-values where the graphs intersect)
b. profit, since the revenue graph is above the cost graph (i.e., the revenue exceeded the cost)
c. loss, since the cost graph is above the revenue graph (i.e., the cost exceeded the revenue)

10. $w = 2$ and $w = 6$ (found by looking where the profit is zero; that is, where the profit graph crosses the w-axis, which is where $P = 0$)

11. To find the break-even points, set each profit formula to zero:
a. $18w - 810 = 0 \Rightarrow 18w = 810 \Rightarrow w = 45$
b. $2.5w - 300 = 0 \Rightarrow 2.5w = 300 \Rightarrow w = 120$
c. $w = 55$
d. $w = 54$

12. $w = 5$. Why is this the break-even point? First, it's where $R = C$; second, it's where $P = 0$.

13. 7 widgets **14**. $P = 4w + 18$ **15**. 99 widgets

16. $w \approx 2.5$; it's the w-coordinate of the point of intersection

“Next in importance
to freedom and justice
is *education*, without
which neither
freedom nor justice
can be permanently
maintained.”

James A. Garfield **(1831 - 1881)**
20th U.S. President